

# Study on fuzzy optimization methods based on principal operation and inequity degree<sup>☆</sup>

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## ABSTRACT

Fuzzy optimization is a well-known optimization problem in artificial intelligence, manufacturing and management, so establishing general and operable fuzzy optimization methods are important in both theory and application. In this paper, by distinguishing principal indices and secondary indices, we give a method for comparing fuzzy information based on synthesizing effect and an operation for achieving fuzzy optimization based on a principal indices transformation. Further, we propose an axiomatic system for fuzzy inequity degree based on the essence of constraint, and give an instructive metric method for fuzzy inequity degree. Then, by combining with genetic algorithm, we give some fuzzy optimization methods based on principal operation and inequity degree (denoted by BPO&ID-FGA, for short). Finally, we consider the convergence of our algorithm using the theory of Markov chains and analyze its performance through two concrete examples. All these indicate that BPO&ID-FGA can effectively merge decision preferences into the optimization process and that it also possesses better global convergence, so it can be applied to many fuzzy optimization problems.

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## 1. Introduction

Fuzziness is a widespread phenomenon in the real world and is unavoidable in many practical fields. In 1965, Zadeh [1] proposed the concept of fuzzy sets and established fuzzy set theory, which formed the foundation for describing and processing uncertain information. After that, a lot of progress has been made in both theory and application, for instance, see [2–4]. The theory of fuzzy numbers, an important part of fuzzy set theory, is very popular in describing uncertain phenomena in actual problems. It is used in many fields such as fuzzy control, fuzzy optimization, fuzzy data analysis, fuzzy time series etc. In 1981, Goetschel and Voxman studied the topological structures of fuzzy numbers and gave an interval representation theorem for fuzzy numbers [5], which laid the foundation for processing fuzzy information and solving fuzzy optimization problems via analytic methods.

Bellman and Zadeh [6] introduced the aggregation operators combining the fuzzy goals and the fuzzy decision space. This paper provided a big boost to the development of fuzzy optimization. Since then, a lot of progress has been made in the study of the fuzzy optimization problems. Here are some interesting references: [7–10] used possibility distribution, [11–13] used fuzzified constraints and objective functions, [14] used fuzzy parameters, and [15] used the embedding theorem, [16–20] transformed a fuzzy linear optimization problem to a classical one by using the structural properties of fuzzy numbers. With

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the development of computer science and evolutionary computation theory, evolutionary computation methods came into play in fuzzy optimization problems. For instance, genetic algorithms were used to deal with optimization problems with fuzzy coefficients but real variables in [21] and [22], and evolutionary computation were used in fuzzy linear optimization problems with fuzzy coefficients and fuzzy variables in [23]. The three papers above have one thing in common: the fuzzy linear optimization problems dealt with in these papers can be transformed into ordinary optimization problems. Up to now, there is no effective method for general fuzzy optimization problems where the main difficulty involves the following aspects: (1) the ordering of fuzzy information; (2) the judgment of fuzzy constraints; (3) the operable description of fuzzy information; (4) the operation of optimization processes.

Many authors made useful explorations in the ordering of fuzzy information from different angles [24–30], and very useful and systematic results have been obtained, but not much progress has been made in the other three aspects. In this paper we deal with the general optimization problems with fuzzy coefficients, fuzzy variables and fuzzy constraint. Our main contributions are: (1) by distinguishing principal indices and secondary indices, we give a comparison method for fuzzy information based on synthesizing effect and a description method for fuzzy information based on principal indices; (2) by using the structural characteristics of fuzzy information and the essence of constraints, we propose an axiomatic system for the fuzzy inequity degree and give an instructive metric method for measuring the fuzzy inequity degree; (3) we establish a very broad and operable fuzzy optimization model and propose a new kind of fuzzy genetic algorithm based on principal operation and inequity degree (denoted by BPO&ID-FGA, for short); (4) we spell out the concrete implementation steps and the operational strategy for crossovers and mutations; (5) we prove the global convergence under the elitist preserving strategy using Markov chain theory; (6) and we further analyze the performance of BPO&ID-FGA by two concrete examples.

## 2. Preliminaries

In the following, let  $R$  be the set of real numbers,  $F(R)$  the family of all fuzzy sets over  $R$ ,  $\int_a^b f(x)dx$  the Lebesgue integral of the function  $f$  on interval  $[a, b]$ . For any  $A \in F(R)$ , the membership function of  $A$  is written as  $A(x)$ , the  $\lambda$ -cuts of  $A$  as  $A_\lambda = \{x | A(x) \geq \lambda\}$ , and the support set of  $A$  as  $\text{supp } A = \{x | A(x) > 0\}$ .

### 2.1. Concept of fuzzy number

Fuzzy numbers, which possess the features of both fuzzy sets and numbers, are the most common tool for describing fuzzy information in real problems. In the following, we recall the definition of fuzzy numbers.

**Definition 2.1** (See [5]).  $A \in F(R)$  is called a fuzzy number if it satisfies the following conditions: (1) for any given  $\lambda \in (0, 1]$ ,  $A_\lambda$  is a closed interval; (2)  $A_1 = \{x | A(x) = 1\} \neq \emptyset$ ; (3)  $\text{supp } A$  is bounded. The class of all fuzzy numbers is called the fuzzy number space, which is denoted by  $E^1$ . In particular, if there exist  $a, b, c \in R$  such that  $A(x) = (x-a)/(b-a)$  for each  $x \in [a, b]$ , and  $A(b) = 1$ , and  $A(x) = (x-c)/(b-c)$  for each  $x \in (b, c]$ , and  $A(x) = 0$  for each  $x \in (-\infty, a) \cup (c, +\infty)$ , then we say that  $A$  is a triangular fuzzy number, and denote it by  $A = (a, b, c)$  for short.

Obviously, if we regard a real number  $a$  as a fuzzy set whose membership function is  $a(a) = 1$  and  $a(x) = 0$  for each  $x \neq a$ , then fuzzy numbers can be thought of as an extension of the real numbers. For any  $A \in E^1$ , the closure of  $\text{supp } A$  is closed interval, in what follows we denote the closure of  $\text{supp } A$  by  $A_0$ .

### 2.2. Operations of fuzzy number

The operations of fuzzy numbers are the foundation for optimization problems. By using the extension principle of fuzzy sets and the properties of fuzzy numbers, we have the following conclusions:

**Theorem 2.1** (See [31]). Let  $A, B \in E^1$ ,  $k \in R$ ,  $f(x, y)$  be a continuous binary function,  $f(A, B)$  be a fuzzy set on  $R$ , and its membership function is  $f(A, B)(z) = \sup\{\min(A(x), B(y)) | x, y \in R, \text{ and } f(x, y) = z\}$ ,  $A_\lambda = [\underline{a}(\lambda), \bar{a}(\lambda)]$ ,  $B_\lambda = [\underline{b}(\lambda), \bar{b}(\lambda)]$  be the  $\lambda$ -cuts of  $A$  and  $B$ , respectively, then  $f(A, B) \in E^1$ , and for any  $\lambda \in [0, 1]$ , we have  $(f(A, B))_\lambda = f(A_\lambda, B_\lambda) = \{f(x, y) | x \in A_\lambda, y \in B_\lambda\}$ . In particular, we have:

- (1)  $A + B = B + A$ ,  $A \cdot B = B \cdot A$ ,  $k(A \pm B) = kA \pm kB$ ;
- (2) If  $A = (a_1, b_1, c_1)$ ,  $B = (a_2, b_2, c_2)$ , then  $A + B = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$ ,  $A - B = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$ ;
- (3) For  $A = (a_1, b_1, c_1)$ , if  $k \geq 0$ , then  $kA = (ka_1, kb_1, kc_1)$ ; if  $k < 0$ , then  $kA = (kc_1, kb_1, ka_1)$ .

Fuzzy numbers have many good analytical properties. For details, please see Ref. [31].

## 3. Compound quantification description of fuzzy information

### 3.1. Basic idea of compound quantification

Ordering fuzzy numbers, a main part of the theory of fuzzy numbers, is the key to fuzzy optimization problems. Up to now, the usual procedure is to map fuzzy numbers by an appropriate transformation into a real number and thus realize a comparison and ordering of fuzzy numbers.

**Definition 3.1** (See [32]). For an uncertain information  $A$ , if the real number  $a$  (called the principal value of  $A$ ) denotes the centralized quantification value in some sense, and the sequence  $a_1, a_2, \dots, a_s$  denote the secondary quantitative indices describing the connection between  $a$  and  $A$  from different angles, we call the total information constituted by  $a$  and  $a_1, a_2, \dots, a_s$  a compound quantification value, written as  $(a; a_1, a_2, \dots, a_s)$  for short.

Taking a triangular fuzzy number  $A = (a_L, a, a_R)$  for example, if we use  $a$  to describe the centralized position of  $A$ ,  $m(A_0) = a_R - a_L$  the divergence of  $A$ , and  $d(A) = \min\{a - a_L, a_R - a\} / \max\{a - a_L, a_R - a\}$  the position degree of  $A$ , then  $(a; m(A_0), d(A))$  could represent a compound quantification of  $A$  in the sense of symmetry. For a general fuzzy number  $A$ , if we take  $a$  as the midpoint of  $A_1 = \{x | A(x) = 1\}$ ,  $a_i$  the length of  $A_{\lambda_i}$  (here  $\lambda_i = (i-1)/n, i = 1, 2, \dots, n$ ), then  $(a; a_1, a_2, \dots, a_n)$  is a compound quantification of  $A$  in the sense of equal step-size.

### 3.2. Compound quantification based on level effect function

**Definition 3.2.** We call  $L(\lambda) : [0, 1] \rightarrow [a, b] \subset [0, \infty)$  a level effect function, if  $L(\lambda)$  is piecewise continuous and non-decreasing. For  $A \in E^1$ , let

$$I(A) = \frac{1}{L^*} \int_0^1 L(\lambda) M_\theta(A_\lambda) d\lambda, \quad (1)$$

$$CD(A) = \int_0^1 L(\lambda) m(A_\lambda) d\lambda. \quad (2)$$

Here,  $L^* = \int_0^1 L(\lambda) d\lambda$ ,  $M_\theta([a, b]) = a + \theta(b - a)$ ,  $\theta \in [0, 1]$ ,  $m$  is the Lebesgue measure.  $I(A)$  is called the centralized quantification value of  $A$ ,  $CD(A)$  the concentration degree of  $A$ . When  $L^* = 0$ ,  $I(A)$  is defined as the midpoint of  $A_1$ ,  $CD(A)$  is defined as the length of  $A_1$ .

Obviously,  $I(A)$  is the centralized quantification value in the sense of level effect function  $L(\lambda)$  and the risk parameter  $\theta$ , it is the principal index for describing the position of  $A$ , while  $CD(A)$  is a secondary index further describing the reliability of  $I(A)$ , so  $(I(A); CD(A))$  is a compound quantification value of  $A$ .

To realize a comparison of fuzzy information, we will map the compound quantification  $(a; a_1, a_2, \dots, a_s)$  of fuzzy information into a real number through an effect synthesizing function.

For an example, for the maximization optimization problems, when we use a two-dimensional array  $(a; b)$  to express the compound quantification of fuzzy information  $A$ , the effect synthesizing function  $S(a, b)$  should satisfy the following conditions: (1) monotone non-decreasing on  $a$ ; (2) monotone non-increasing on  $b$ ; (3)  $S(a, 0) = a$ . And in the implementation process of algorithm in Section 8, we select  $S(a, b) = a / (1 + \beta b)^\alpha$  as the effect synthesizing function, here both  $\alpha, \beta \in (0, +\infty)$  represent some kind of decision preferences, and their value can be determined through the divergence degree of fuzzy information, for example: when a decision-maker thinks that the divergence degree  $b$  takes 2 and 8, and the reliability of the centralized quantification value is 0.90 and 0.65 respectively, then we can compute the value of  $\alpha, \beta$  through the formulas  $(1 + 2\beta)^{-\alpha} = 0.90$  and  $(1 + 8\beta)^{-\alpha} = 0.65$ .

In practical problems, we can choose different forms of compound quantification of fuzzy information and different synthesizing effect function according to different guiding ideology of decision-making. Such changes do not have any substantial impact on the fuzzy optimization model in Section 5 and the structure of genetic algorithm in Section 6.

## 4. Compound quantification description of fuzzy constraint

Generally, the constraints of fuzzy optimization problems have some uncertainty, therefore a key step is to judge if the constraints are satisfied. So far the most commonly used method is a method based on the order relation of fuzzy information. Due to the essential differences between fuzzy numbers and real numbers, the method mentioned above is not very satisfactory, even if they agree with the general principles of optimization systems. For instance,

(1) The fuzzy order relation based on level cut (that is, for any  $\lambda \in [0, 1]$ ,  $A, B \in E^1$ ,  $A \leq B \Leftrightarrow A_\lambda \leq B_\lambda$ , and  $[a, b] \leq [c, d] \Leftrightarrow a \leq c, b \leq d$ ) cannot be used to solve the judgment problem of fuzzy constraints because this order relation is incomplete.

(2) The fuzzy order relation based on compound quantification (take formula (1) as an example, for  $A, B \in E^1$ , if  $I(A) \leq I(B)$ , then  $A \leq B$ ) is a synthesizing judgment of differences between  $A$  and  $B$  under all levels, since  $I(A) - I(B) = \frac{1}{L^*} \int_0^1 L(\lambda) [M_\theta(A_\lambda) - M_\theta(B_\lambda)] d\lambda$ . In practical problems, we are only concerned with whether or not  $A$  is more than  $B$ , and it is not necessary to know the degree of  $A$  exceeding  $B$  or not, so this method can't completely describe the essence of constraints.

In view of the shortcomings mentioned above, references [33,34] defined the degree  $D(A \leq x)$  of a fuzzy number  $A$  not exceeding a real number  $x$  by the location relationship of all level cuts and  $x$ , then gave the definition of  $D(A \leq B)$ , the degree of a fuzzy number  $A$  not exceeding a fuzzy number  $B$  by  $D(A - B \leq 0)$ . Further, by combining with a given threshold  $\beta \in (0, 1]$ , [33,34] gave a method to tell whether or not  $A \leq B$  by checking whether or not  $D(A \leq B) \geq \beta$ . Since the addition operation and subtraction operation of fuzzy numbers are not inverse of each other, defining the degree of  $A \leq B$  by the degree of  $A - B \leq 0$  is not reasonable. For instance, if  $A_\lambda (0 < \lambda < 1)$  is not a singleton, then  $D(A \leq A) = 0.5$ .

From the above analysis, we see that the commonly used methods of testing fuzzy constraints all have a certain weakness. To establish general rules for processing fuzzy constraints, we introduce an axiomatic system for fuzzy inequity degree as follows:

**Definition 4.1.** Let  $D(A, B)$  be a function on  $E^1 \times E^1$  (denoted by  $D(A \leq B)$  for short).  $D$  is called a fuzzy inequity degree on  $E^1$  if it satisfies:

- (1) *Normality*:  $0 \leq D(A \leq B) \leq 1$  for any  $A, B \in E^1$ ;
- (2) *Reflexivity*:  $D(A \leq A) = 1$  for any  $A \in E^1$ ;
- (3) *Monotonicity*:  $D(A^{(1)} + A^{(2)} \leq B^{(1)} + B^{(2)}) = 1$  for any  $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)} \in E^1$  with  $D(A^{(1)} \leq B^{(1)}) = D(A^{(2)} \leq B^{(2)}) = 1$ ;
- (4) *Semi-linearity*:  $D(kA \leq kB) = D(A \leq B)$  for any  $A, B \in E^1$  and  $k \in (0, \infty)$ ;
- (5) *Translation invariance*:  $D(a + A, a + B) = D(A, B)$  for any  $A, B \in E^1$  and  $a \in \mathbb{R}$ .

In Definition 4.1 1 and 0 denote absolute satisfaction state and dissatisfaction state respectively. Obviously, the different requirements above reflect the basic characteristic of the “no excess” relationship from different aspects.

For given  $\alpha \in [0, 1]$ , let

$$D(A \leq B) = H(M_\theta(B_\alpha) - M_\theta(A_\alpha)). \quad (3)$$

Here,  $M_\theta([a, b]) = a + \theta(b - a)$ ,  $\theta \in [0, 1]$ ;  $H(x) = 1$  for each  $x \in [0, +\infty)$ , and  $H(x) = 0$  for each  $x \in (-\infty, 0)$ . It is easy to verify using Definition 4.1 that formula (3) is a fuzzy inequity degree on  $E^1$ .

From formula (3) we know that this kind of fuzzy inequity degree contains the “no excess” relationship  $\leq$ , but it doesn't make full use of the location relationship of  $A$  and  $B$  under all levels. To establish a better model for describing the fuzzy inequity degree, we introduce the following formula (4):

$$D(A \leq B) = \frac{1}{L^*} \int_0^1 L(\lambda) H(M_\theta(B_\lambda) - M_\theta(A_\lambda)) d\lambda. \quad (4)$$

Here,  $L(\lambda)$  is the level effect function,  $L^* = \int_0^1 L(\lambda) d\lambda$ ; and if  $L^* = 0$ , then  $D(A \leq B) = H(M_\theta(B_1) - M_\theta(A_1))$ .

**Theorem 4.1.**  $D(A \leq B)$  defined by formula (4) is a fuzzy inequity degree on  $E^1$ .

**Proof.** It is clear that, if  $L^* = 0$ , then  $D(A \leq B) = H(M_\theta(B_1) - M_\theta(A_1))$  is a fuzzy inequity degree on  $E^1$ . In the following, we suppose that  $L^* \neq 0$ .

(1) For any  $\theta \in [0, 1]$  and  $A, B \in E^1$ ,  $\lambda \in [0, 1]$ , using  $0 \leq H(M_\theta(B_1) - M_\theta(A_1)) \leq 1$ ,  $H(M_\theta(A_\lambda) - M_\theta(A_\lambda)) \equiv 1$ , we know that  $D(A \leq B)$  satisfies the conditions (1) and (2) in Definition 4.1.

(2) For any given  $k \in (0, +\infty)$ , as  $k[a, b] = [ka, kb]$ , we have that  $M_\theta(k[a, b]) = kM_\theta([a, b])$ . It follows from this and Theorem 2.1 that

$$M_\theta((kB)_\lambda) - M_\theta((kA)_\lambda) = M_\theta(kB_\lambda) - M_\theta(kA_\lambda) = k(M_\theta(B_\lambda) - M_\theta(A_\lambda))$$

for any  $\lambda \in [0, 1]$ ,

therefore  $M_\theta((kB)_\lambda) - M_\theta((kA)_\lambda)$  and  $M_\theta(B_\lambda) - M_\theta(A_\lambda)$  have the same sign. Using this and the definition of  $H(x)$ , we have  $D(kA \leq kB) = D(A \leq B)$ , that is,  $D(A \leq B)$  satisfies the condition (4) in Definition 4.1.

(3) Since  $a + [b, c] = [a + b, a + c]$ , we have  $M_\theta(a + [b, c]) = a + M_\theta([b, c])$ , together with Theorem 2.1, we know that  $M_\theta((a+B)_\lambda) - M_\theta((a+A)_\lambda) = [a + M_\theta(B_\lambda)] - [a + M_\theta(A_\lambda)] = M_\theta(B_\lambda) - M_\theta(A_\lambda)$  for any  $\lambda \in [0, 1]$ , therefore  $D(a+A, a+B) = D(A, B)$ , that is,  $D(A \leq B)$  satisfies the condition (5) in Definition 4.1.

(4) Since  $L^* \neq 0$ , there must exist  $K \subset [0, 1]$  such that  $m(K) > 0$ , and  $L(\lambda) > 0$  for each  $\lambda \in K$ ,  $L(\lambda) = 0$  for each  $\lambda \in [0, 1] - K$ . For simplicity, we assume that  $L(\lambda) > 0$  for any  $\lambda \in [0, 1]$ . In the following, we prove that  $D(A \leq B)$  satisfies the condition (3) in Definition 4.1.

In fact, by  $M_\theta([a, b] + [c, d]) = M_\theta([a, b]) + M_\theta([c, d])$  and  $D(A \leq B) = 1 \Leftrightarrow \int_0^1 L(\lambda) H(M_\theta(B_\lambda) - M_\theta(A_\lambda)) d\lambda = \int_0^1 L(\lambda) d\lambda \Leftrightarrow H(M_\theta(B_\lambda) - M_\theta(A_\lambda)) = 1 \Leftrightarrow M_\theta(B_\lambda) - M_\theta(A_\lambda) > 0$ , we can obtain that the following relations:

$$\begin{aligned} D(A^{(1)} \leq B^{(1)}) &= D(A^{(2)} \leq B^{(2)}) = 1 \\ \Leftrightarrow M_\theta(B_\lambda^{(1)}) - M_\theta(A_\lambda^{(1)}) > 0 \quad \text{and} \quad M_\theta(B_\lambda^{(2)}) - M_\theta(A_\lambda^{(2)}) > 0 &\text{almost everywhere on } [0, 1] \\ \Rightarrow M_\theta(B_\lambda^{(1)}) + M_\theta(B_\lambda^{(2)}) - M_\theta(A_\lambda^{(1)}) - M_\theta(A_\lambda^{(2)}) > 0 &\text{almost everywhere on } [0, 1] \\ \Leftrightarrow M_\theta(B_\lambda^{(1)} + B_\lambda^{(2)}) - M_\theta(A_\lambda^{(1)} + A_\lambda^{(2)}) > 0 &\text{almost everywhere on } [0, 1] \\ \Leftrightarrow M_\theta((B^{(1)} + B^{(2)})_\lambda) - M_\theta((A^{(1)} + A^{(2)})_\lambda) > 0 &\text{almost everywhere on } [0, 1] \\ \Leftrightarrow D(A^{(1)} + A^{(2)} \leq B^{(1)} + B^{(2)}) &= 1. \quad \square \end{aligned}$$

In the optimization and decision process of many real problems, the degree of importance to the studied problems varies with different levels, so the influence of the degree of  $A_\lambda \leq B_\lambda$  at different levels on the global degree of  $A \leq B$  is not the same. In formula (4), the level effect function  $L(\lambda)$  is a kind of decision parameter describing the effect value under different levels, therefore, formula (4) is essentially an instructive metric method reflecting the fuzzy information  $A$  not exceeding  $B$ . In practical problems, we can select the risk parameter  $\theta$  and the level effect function  $L(\lambda)$  according to different decision preferences.

## 5. The solution model of fuzzy optimization problem based on inequity degree

### 5.1. Formalized description of fuzzy optimization problems

The general form of an optimization problem is to find the maximum (or minimum) of certain “objective function” under some “constraints”; the solution to this problem is known as the optimization method. In classical optimizations, the objective function and constraints are deterministic (i.e., not fuzzy). In practice, however, the objective function as well as the constraint conditions often have uncertainty in different forms, so solving uncertain optimization problems is important in both theory and applications. In this paper we will consider the following optimization problems in which both objective function and the constraints are with fuzzy uncertainty, the general form of our mathematical model can be expressed as:

$$\begin{cases} \max f(x), \\ \text{s.t. } g_i(x) \leq b_i, \quad i = 1, 2, \dots, m, \end{cases} \quad (5)$$

here,  $x = (x_1, x_2, \dots, x_n)$ ,  $f$  and  $g_1, g_2, \dots, g_m$  are  $n$ -dimensional fuzzy value functions (namely  $n$ -dimensional function with fuzzy value),  $\leq$  denotes the inequality relationship in the fuzzy sense,  $x_i \in E^1$  the optimized variables or decision variables,  $b_i \in E^1$  the given fuzzy numbers.

### 5.2. The fuzzy optimization model based on inequity degree

Because fuzzy numbers do not have a complete order like the real numbers, model (5) is just a formal model, and can't be easily solved. Using the compound quantification strategy and the fuzzy inequity degree above, it can be converted into the following model (6) by the synthesizing effect function:

$$\begin{cases} \max E(f(x)), \\ \text{s.t. } D(g_i(x) \leq b_i) \geq \beta_i, \quad i = 1, 2, \dots, m. \end{cases} \quad (6)$$

Here,  $E(f(x))$  denotes the synthesizing effect value of the fuzzy value function  $f(x)$ ,  $D(g_i(x) \leq b_i)$  denotes the degree of  $g_i(x) \leq b_i$ ,  $\beta_i \in (0, 1]$  denotes the minimum requirement for satisfying  $g_i(x) \leq b_i$ . If formula (1) and formula (2) are taken as the compound quantification description of fuzzy information,  $S(a, b)$  as the synthesizing effect operator, formula (4) as a measure of inequity degree, then we have

$$E(f(x)) = S(I(f(x))), CD(f(x)), D(g_i(x) \leq b_i) = \frac{1}{L^*} \int_0^1 L(\lambda) H(M_\theta(g_i(x)_\lambda) - M_\theta((b_i)_\lambda)) d\lambda.$$

Obviously, model (6) have the features of an optimization operation, but it is not a conventional optimization problem, and can't be solved by the existing methods. The main difficulty is that it is hard to describe in detail the way of fuzzy information changes. Since triangular fuzzy numbers are often used to describe fuzzy information in practical problems, we will assume in this paper that the optimized variables and the coefficients are all triangular fuzzy numbers. Using the structural feature of fuzzy numbers and the density of step type fuzzy numbers and quasi-linear fuzzy numbers in the fuzzy number space (see Ref. [35]), a general solution operation can be established for fuzzy optimization problems with general fuzzy variables, which will be discussed in a forthcoming paper.

Due to the intrinsic difference between the operations of triangular fuzzy numbers and those of the real numbers, optimization problems with triangular fuzzy variables and triangular fuzzy coefficients cannot be solved by analytical methods even though triangular fuzzy numbers are much easier to describe than general fuzzy numbers. We establish concrete solution methods to our optimization problem by combining genetic algorithm with the compound quantification strategy of fuzzy information (denoted by BPO&ID-FGA, for short).

## 6. Fuzzy genetic algorithm based on principal operation and inequity degree (BPO&ID-FGA)

As a random searching optimization method simulating natural evolutionary process, genetic algorithms [36] possess the features of easy operation and strong flexibility, thus genetic algorithms are one of the most commonly used methods in the fields of intelligent computation and optimization of complex systems. In this section, we will focus on the structure of BPO&ID-FGA. The basic operation strategies are described from the following three aspects:

(1) For the decision variable  $A = (a, b, c)$ , we view  $b$  as the principal index describing the size position of  $A$ ,  $a$  and  $c$  the secondary indices. In the optimization process, we first consider the change of the principal index  $b$ , and then by combining the lengths of  $[a, b]$  and  $[b, c]$  with the new value of  $b$ , we determine the new values of the secondary indices  $a$  and  $c$  by some strategy. The fact that the new value  $A' = (a', b', c')$  of  $A = (a, b, c)$  depends mainly on the principal index  $b$  in this kind of operational strategy is one of the main reasons why we named our algorithm a genetic algorithm based on principal operation and inequity degree.

(2) For the evaluation of the objective function, we take the synthesizing effect value of the compound quantification description of fuzzy information described by formula (1) and formula (2) as the main criterion of operation. From our

discussions in Section 3, we know that the main things involved in this process are the concepts of principal index and secondary indices. This is the second main reason why we named our algorithm as we did.

(3) For the satisfaction of the fuzzy constraints, we take the fuzzy inequity degree (4) as the main criterion; this is the third main reason why we named our algorithm as we did.

Since in applications the value of the objective function is nonnegative, we assume in the following that: (1)  $E(f(x)) \geq 0$ , otherwise, we can convert it into  $M + E(f(x))$  by selecting an appropriate large  $M$ ; (2) The optimization problem is an maximization one, and the minimization problem  $\min f(x)$  can be converted into a maximization problem by  $\max[M - E(f(x))]$ , where,  $M$  is an appropriate large positive number.

### 6.1. Coding

Coding is the most basic component of genetic algorithm. And among the numerous methods, binary coding is the most commonly used one. Binary coding is simple to operate and easy to analyze theoretically. In BPO&ID-FGA, for fuzzy number  $(a, b, c)$ , we adopted three 0, 1 codes of equal length to respectively represent the principal index  $b$  and the left, right secondary indices  $a$  and  $c$ .

### 6.2. Crossover and mutation

The crossover and mutation operators are the specific strategies used to find the optimal or satisfactory solution. In BPO&ID-FGA, we only apply the crossover and mutation operations to the middle section of the chromosome which represents the principal index of the fuzzy variable. And we obtain the two ends of the coding string by some strategy. The details are given below.

#### 6.2.1. Crossover operation

For two given fuzzy numbers  $A^{(1)} = (a_1, b_1, c_1)$  and  $A^{(2)} = (a_2, b_2, c_2)$ , cross the two strings representing  $b_1$  and  $b_2$ , respectively, and take one of the resulting strings  $b$  as the crossover result of  $b_1$  and  $b_2$ , then determine the left and right secondary indices  $a$  and  $c$  by some strategy. Generally speaking, we can use the following methods to determine  $a$  and  $c$  (here, both  $r_1$  and  $r_2$  are random numbers in a specified region):

- ①  $a = b - r_1 b, c = b + r_2 b$ ;
- ②  $a = b - r_1, c = b + r_2$ ;
- ③  $a = b - r_1(b_1 - a_1) - r_2(b_2 - a_2), c = b + r_1(c_1 - b_1) + r_2(c_2 - b_2)$ .

#### 6.2.2. Mutation operation

For any given fuzzy number  $A = (a, b, c)$ , mutate the string representing  $b$  to obtain a mutated result  $b'$ , then determine the left and right secondary indices  $a'$  and  $c'$  of  $b'$  by some strategy. Usually, we can use the following methods to determine  $a'$  and  $c'$  (here, both  $r_1$  and  $r_2$  are random numbers in a specified region):

- ①  $a' = b' - r_1 b', c' = b' + r_2 b'$ ;
- ②  $a' = b' - r_1, c' = b' + r_2$ ;
- ③  $a' = b' - r_1(b - a), c' = b' + r_1(c - b)$ .

In the process above, the ranges of variation of the random number  $r_1$  and  $r_2$  may be different, and they vary with our ways to determine the left and right secondary indices. In this paper, we choose ③s as the method of crossover and mutation.

In practical fuzzy optimization problems, the decision maker is mainly concerned with the size of principal index which represents the position of the fuzzy information, thus it is reasonable for us to search the optimal solution starting from the principal index.

### 6.3. Replication

In designing a genetic algorithm, the penalty strategy is commonly used to eliminate constraints in the optimization process. Its purpose is to convert infeasible solutions into a feasible solution by adding a penalty item in the objective function which lowers the possibility of an infeasible solution being selected for evolution. In BPO&ID-FGA, we use the following fitness function with some penalty strategy:

$$F(x) = E(f(x)) \cdot p(x). \quad (7)$$

And we take formula (7) as the basis of proportional selection. Here,  $E(f(x))$  is the synthesizing effect value of the objective function  $f(x)$ ,  $p(x)$  is the penalty factor, the basic form of which is as follows:

- (1) If all the constraints are satisfied, then  $p(x) = 1$ ;
- (2) If the constraints are not completely satisfied, then  $0 \leq p(x) \leq 1$ .

In general, the exponential function can be used as a penalty function as follows:

$$p(x) = \exp \left\{ -K \cdot \sum_{i=1}^m \alpha_i \cdot r_i(x) \right\}. \quad (8)$$



Here,  $K \in (0, \infty]$  is a parameter reflecting the penalty intensity,  $\alpha_i \in (0, \infty]$  is a grade parameter of the  $i$ th constraint,  $r_i(x) \in [0, \infty)$  is a parameter describing the degree that the  $i$ th constraint is violated. In this paper we use the convention  $0 \cdot \infty = 0$ .

Obviously,  $K = \infty$  implies that the decision result must satisfy all the constraints,  $\alpha_i = \infty$  implies that the decision result must satisfy the  $i$ th constraints, and  $0 < \alpha_i, K < \infty$  implies that the decision result can break  $i$ th constraint to some degree. In the example below, we will take  $\alpha_i = 1, K = 0.01, r_i(x)$  to be the degree that the  $i$ th constraints is violated (namely, the difference of synthesizing effect value between two sides of constraints),  $i = 1, 2, \dots, m$ .

## 7. Convergence of BPO&ID-FGA

We know from the discussions above that the crossover, mutation and selection in BPO&ID-FGA only depend on the current state of population; they have nothing to do with the earlier states. Thus the BPO&ID-FGA is a Markov chain and its convergence could be analyzed by the Markov chain theory.

**Definition 7.1** (See [37]). Let  $X(n) = \{X_1(n), X_2(n), \dots, X_N(n)\}$  be the  $n$ th population of genetic algorithm,  $Z_n$  denote the optimal value in the population  $X(n)$ , that is,  $Z_n = \max\{f(X_i(n)) | i = 1, 2, \dots, N\}$ . If  $\lim_{n \rightarrow \infty} P\{Z_n = f^*\} = 1$ , then we say the genetic sequence  $\{X(n)\}_{n=1}^\infty$  converges. Here,  $f^* = \max\{f(X) | X \in S\}$  denotes the global optimal value of the individuals.

**Lemma 7.1.** *The genetic sequence  $\{X(t)\}_{t=1}^\infty$  of BPO&ID-FGA is a homogeneous Markov chain which is mutually attainable.*

**Proof.** (1) We first verify that the genetic algorithm with fuzzy fitness value is a homogenous Markov chain. Let  $K$  be the size of the population space with binary encoding,  $S = \{s_1, s_2, \dots, s_K\}$  be the finite state space composed by all the possible populations, then  $K = 2^{N-1}$ . We know from the operating process of BPO&ID-FGA that the  $(t+1)$ th population  $X(t+1)$  is obtained from the  $t$ th population  $X(t)$  under some genetic operation in a certain probability, and it is independent of  $X(t-1), X(t-2), \dots, X(1), X(0)$ . Therefore

$$P(X(t+1) = s_{i_{t+1}} | X(0) = s_{i_0}, X(1) = s_{i_1}, \dots, X(t) = s_{i_t}) = P(X(t+1) = s_{i_{t+1}} | X(t) = s_{i_t}).$$

Here,  $P(X(t+1) = s_{i_{t+1}} | X(t) = s_{i_t})$  denotes the conditional probability of state  $X(t+1) = s_{i_{t+1}}$ , and  $i_1, i_2, \dots, i_t, i_{t+1}$  are all positive integers; this implies that  $\{X(t)\}_{t=1}^\infty$  is a Markov chain.

Let  $p_{ij}^{(n)}(t) = P(X(t+n) = s_j | X(t) = s_i)$  denote the transition probability of state  $s_i$  to  $s_j$  after  $n$  steps in  $t$ th population. Because the transition probability of each generation in BPO&ID-FGA only depends on the crossover probability, the mutation probability as well as the population of this generation, and it does not change with time (e.g. evolution generation), that is,  $p_{ij}^{(n)}(t)$  is independent of the initial time  $t$ , so  $\{X(t)\}_{t=1}^\infty$  is a homogenous Markov chain.

(2) Let  $P = (p_{ij}^{(1)})_{K \times K}$  be the one-step transition probability matrix of  $\{X(t)\}_{t=1}^\infty$ . In what follows we prove that  $\{X(t)\}_{t=1}^\infty$  is mutually attainable.

For two binary strings  $b = b_1 b_2 \dots b_l$  and  $b' = b'_1 b'_2 \dots b'_l$  with length  $l$ , let

$$H(b, b') = \sum_{i=1}^l |b_i - b'_i| \quad (9)$$

be the Hamming distance between  $b$  and  $b'$ . Then for any given mutation probability  $p_m \in (0, 1)$ , the probability  $P(b \rightarrow b')$  of transferring from  $b$  to  $b'$  is

$$P(b \rightarrow b') = p_m^{1-H(b,b')} (1 - p_m)^{H(b,b')} > 0. \quad (10)$$

We can see from formula (10) that if the mutation is the only operation, then any two binary strings with the same length are mutually attainable (that is,  $P(b' \rightarrow b) > 0$  also holds). If we only consider the mutation operation in BPO&ID-FGA, then we know from the analysis above that the one-step transition probability between states  $s_i$  and  $s_j$  is bigger than 0, which implies that the  $n$ -step transition probability between states  $s_i$  and  $s_j$  will be bigger than 0 as well, that is  $p_{ij}^{(n)} > 0$  and  $p_{ji}^{(n)} > 0$  from Chapman–Kolmogorov equation. Because selection and crossover are all operated under a certain probability sense, the two operations do not affect the mutual attainability between two states, which indicates that in BPO&ID-FGA the transition probability starting from any state, passing through any finite steps, to another state is bigger than 0, which is to say that  $\{X(t)\}_{t=1}^\infty$  is a mutually attainable Markov chain.  $\square$

**Lemma 7.2.** *The genetic sequence  $\{X(t)\}_{t=1}^\infty$  of BPO&ID-FGA is an ergodic Markov chain.*

**Proof.** We see from Lemma 7.1 that the genetic sequence  $\{X(t)\}_{t=1}^\infty$  of BPO&ID-FGA is an irreducible, positive recurrent and non-periodic Markov chain. From the theory of Markov chains we see that BPO&ID-FGA is an ergodic Markov chain, and its stationary probability distribution exists, that is, there exist  $p_j > 0, j = 1, 2, \dots, K$  such that  $p_{ij}^{(n)} \rightarrow p_j (n \rightarrow \infty)$  and  $\sum_{j=1}^K p_j = 1$ , and which is independent of the original states.  $\square$

**Table 1**Approximate units of food  $F_j$  and product  $P_i$ 

	$P_1$	$P_2$	$P_3$
$F_1$ (1 unit)	About 2.5 g	About 4.5 g	About 5 g
$F_2$ (1 unit)	About 5 g	About 3 g	About 10 g

**Table 2**

The specific description of Table 1

	$P_1$	$P_2$	$P_3$
$F_1$ (1 unit)	(2, 2.5, 3)	(4, 4.5, 5)	(4.5, 5, 5.5)
$F_2$ (1 unit)	(4.5, 5, 5.5)	(2.5, 3, 3.5)	(9, 10, 11)

**Theorem 7.1.** The BPO&ID-FGA using the elitist preserving strategy in replication process (that is, the contemporary optimal individual persevered to the next generation) is globally convergent.

**Proof.** Denote  $p_{ij}^{(n)}$  to be transition probability of state  $i$  to  $j$  after  $n$  steps. Suppose the state of contemporary population (for example generation  $t$ ) is  $j$ , because the elitist preserving strategy is used, then some individual in state  $j$  of generation  $t$  (for instance the individual at position  $k$ ) is the most superior individual of the previous generation (generation  $t - 1$ ), which indicates that the most superior individual of generation  $t$  is more excellent than or equal to that of generation  $t - 1$ .  $\square$

We suppose that generation  $t'$  before generation  $t$ , and  $i$  be the population state of generation  $t'$ , and a more superior new individual is produced in the evolution process from generation  $t'$  to generation  $t$  (namely the most superior individual of generation  $t$  is more outstanding than the most superior individual of generation  $t'$ ). It is very obvious that  $p_{ij}^{(n)} > 0$  from the ergodic of mutation by now, which is to say it is reachable from  $i$  to  $j$ . Simultaneously we also obtain that  $p_{ji}^{(n)} = 0$  from the properties of elitist preserving strategy, which is to say that it is inaccessible from  $j$  to  $i$ . In the analysis above, for  $i$  and  $j$  arbitrary, we may obtain that: BPO&ID-FGA using the elitist preserving strategy is a irreversible evolution process, and it will finally converge to the global optimal solution.

## 8. Application examples

**Example 1** (Diet Problem [13]). A farmer has three products  $P_1$ ,  $P_2$  and  $P_3$  which he plans to mix together to feed his pigs. He knows that the pigs require a certain amount of food  $F_1$  and  $F_2$  per day. Table 1 presents the estimates of the units of  $F_1$  and  $F_2$  available, per gram of  $P_1$ ,  $P_2$  and  $P_3$ . Also, each pig should have approximately at least 54 units of  $F_1$  and approximately at least 60 units of  $F_2$ , per day. The costs of  $P_1$ ,  $P_2$  and  $P_3$  vary slightly from day to day but the average costs are: (1) 8 cents per gram of  $P_1$ ; (2)  $P_2$  is 9 cents per gram; and (3) 10 cents /gram for  $P_3$ . The farmer wants to know how many grams of  $P_1$ ,  $P_2$  and  $P_3$  he should mix together each day, so his pigs will get the approximate minimum, to minimize his costs.

This problem is evidently an uncertain optimization problem. If the term “about  $c$ ” is interpreted as a fuzzy number with membership degree 1, and we define respectively the cost of three products as (7, 8, 9), (9, 10, 11) and (10, 11, 12), then Table 1 can be converted into the following Table 2:

Then the above solution scheme can be converted to the following fuzzy linear optimization model:

$$\begin{aligned}
 \min \quad & f(x_1, x_2, x_3) = (7, 8, 9)x_1 + (8, 9, 10)x_2 + (9, 10, 11)x_3 \\
 \text{s.t.} \quad & (2, 2.5, 3)x_1 + (4, 4.5, 5)x_2 + (4.5, 5, 5.5)x_3 \gtrsim (50, 54, 58) \\
 & (4.5, 5, 5.5)x_1 + (2.5, 3, 3.5)x_2 + (9, 10, 11)x_3 \gtrsim (56, 60, 64) \\
 & x_1, x_2, x_3 \gtrsim 0.
 \end{aligned}$$

For this optimization problems (both coefficients and variables are real numbers), the optimal solutions are  $x_1 = 0$ ,  $x_2 = 8$ ,  $x_3 = 3.6$  and  $\min f(x_1, x_2, x_3) = 108$  cents.

Let the size of the population be 80, (1) be the centralized quantification value, (2) be the concentration degree of  $A$ ,  $S(I(A), CD(A)) = I(A)/[1 + 0.0001 \cdot CD(A)]^{0.5}$  be the synthesizing effect function, and  $L(\lambda) = \lambda$ . By using BPO&ID-FGA with 30 bits of binary coding, we can get the minimum value shown on Fig. 1 after 100 times of iterations (taking the times of iteration as  $x$ -coordinate, and the synthesizing effect value of fuzzy minimum value as  $y$ -coordinate). The optimal solutions are  $x_1 = (0, 0, 0.1242)$ ,  $x_2 = (7.7355, 8.0000, 8.0706)$  and  $x_3 = (3.2927, 3.6000, 3.9271)$ . And the synthesizing effect value of the fuzzy minimum value is 108.1002.

We see from the above that the optimization results are very close to those in [13], but the solution process and the method are superior to that in [13]. We need not to transform it into a multi-objective programming by some strategy. Through the method in this paper, we cannot only deal with the programming problems with fuzzy variables, but also nonlinear programming problems, and this method possesses the features of easy operation and wide application; at the same time, it can effectively merge decision preferences into decision processes. In the following, we will further analyze the performance of BPO&ID-FGA in nonlinear programming problems.



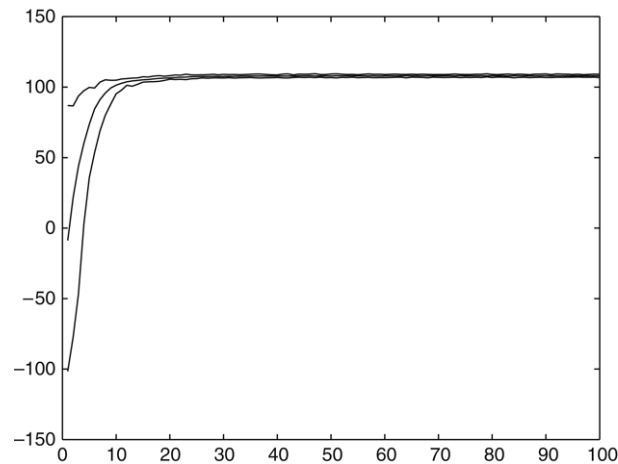


Fig. 1. 100 iteration results for Example 1.

Table 3

Computation results of Test 1

	$(\alpha, \beta)$	Optimization solutions	$Y_1$	$Y_2$	CD	CT	C
Case 1	(0.5, 0.1)	$x_1 = (4.7628, 5.0000, 5.3213), x_2 = (10.7847, 10.9785, 11.2183)$	224.5967	137.9064	9.9930	21.5160	21
Case 2	(0.5, 1)	$x_1 = (4.9370, 5.0000, 5.1036), x_2 = (10.8373, 11.0000, 11.0212)$	224.2314	49.4265	9.2659	18.8130	22
Case 3	(2, 0.1)	$x_1 = (4.5064, 4.9756, 5.0580), x_2 = (8.4635, 8.7527, 9.0867)$	201.4342	42.0757	7.8990	18.6250	21
Case 4	(2, 1)	$x_1 = (1.8102, 2.2385, 2.7164), x_2 = (3.2860, 3.3118, 3.6266)$	92.2965	2.4456	3.2300	20.7970	19

**Example 2.** The fuzzy nonlinear programming problem is as follows

$$\begin{aligned}
 \max \quad & f(x_1, x_2) = -(0.1, 0.3, 0.8)x_1^2 - (0.2, 0.4, 0.7)x_2^2 + (16.1, 17, 17.3)x_1 + (17.7, 18, 18.6)x_2, \\
 \text{s.t.} \quad & (1.4, 2, 2.6)x_1 + (2.7, 3, 3.3)x_2 \leq (47, 50, 51), \\
 & (3.8, 4, 4.4)x_1 + (1.6, 2, 2.2)x_2 \leq (40, 44, 47), \\
 & (2.6, 3, 3.2)x_1 + (1.6, 2, 2.2)x_2 \leq (32, 36, 40), \\
 & x_1, x_2 \geq 0.
 \end{aligned}$$

For this optimization problem (both coefficients and variables are real numbers), the optimal solutions are  $x_1 = 4.8333, x_2 = 10.75, \max f(x_1, x_2) = 222.4329$ .

Let the size of the population be 80, (1) be the centralized quantification value, (2) be the concentration degree of  $A$ ,  $S(I(A), CD(A)) = I(A)/[1 + 0.001 \cdot CD(A)]^{0.5}$  be the synthesizing effect function,  $L(\lambda) = \lambda$ . By using BPO&ID-FGA with 20 bits of binary coding, we can get the optimal value shown on Fig. 2 after 100 times of iterations (taking the times of iteration as  $x$ -coordinate, and the synthesizing effect value of fuzzy minimum value as  $y$ -coordinate). The optimal solutions are  $x_1 = (4.6595, 4.9902, 5.3576), x_2 = (10.5398, 11.0000, 11.4577)$ , and the synthesizing effect value of fuzzy maximum value is 222.1152.

In order to further analyze the performance of BPO&ID-FGA, we take different synthesizing effect functions and level effect functions and do the following tests:

**Test 1.** For  $L(\lambda) = \lambda$  and  $S(I(A), CD(A)) = I(A)/(1 + \beta \cdot CD(A))^\alpha$ , and  $(\alpha, \beta)$  takes (0.5, 0.1), (0.5, 1), (2, 0.1) and (2, 1), respectively, the computation results are stated in Table 3.

**Test 2.** For  $S(I(A), CD(A)) = I(A)/(1 + 0.01 \cdot CD(A))^{0.5}$ , and  $L(\lambda)$  be  $\lambda, \lambda^2, \lambda^{0.5}, L(\lambda) = \begin{cases} \lambda & 0.3 \leq \lambda \leq 1 \\ 0 & 0 \leq \lambda < 0.3 \end{cases}$ , respectively, the computation results are stated in Table 4.

**Test 3.** For  $S(I(A), CD(A)) = I(A)/(1 + 0.001 \cdot CD(A))^{0.5}$  and  $L(\lambda) = \lambda$ , the results of 10 experiments are stated in Table 5.

In Tables 3–5,  $Y_1$  denotes the centralized quantification value of the maximum value,  $Y_2$  the synthesizing effect value of the maximum value, CD concentration degree, C the convergence generation, CT denotes computation times, and AV the average value.

All the calculations above are based on Matlab 6.5 and a 2.00 GHz Pentium 4 processor and worked out under a Windows XP Professional Edition platform. From the results above we see that: (1) The computational results are related to the level effect function and the synthesizing effect function, and the difference is obvious (for instance: case 1 and case 4 in Test 1), which shows BPO&ID-FGA can effectively merge decision preferences into the decision process; (2) Despite the variations of parameters, the convergence time is about 20 s, and the convergence generation is about 20; also, the rate of getting the optimal result is almost more than 80%, which shows the algorithm has higher computational efficiency and good

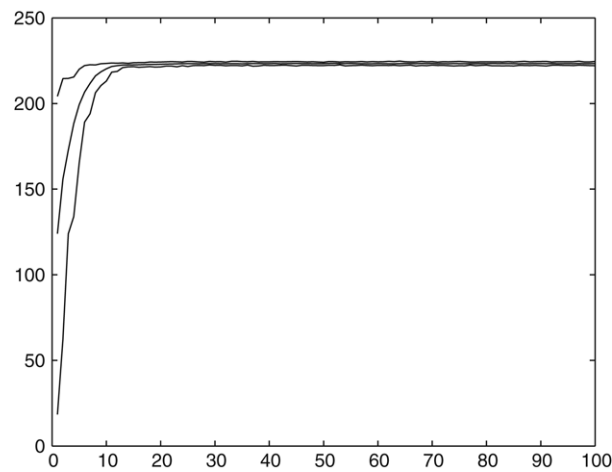


Fig. 2. 100 iteration results for Example 2.

Table 4

Computation results of Test 2

	$L(\lambda)$	Optimization solutions	$Y_1$	$Y_2$	CD	CT	C
Case 1	$\lambda$	$x_1 = (4.9150, 5.0000, 5.3484), x_2 = (10.5883, 11.0000, 11.1770)$	224.8013	213.1663	9.9593	20.5630	14
Case 2	$\lambda^2$	$x_1 = (4.5137, 4.9853, 5.0258), x_2 = (10.6904, 11.0000, 11.4684)$	224.6494	217.2436	6.7096	21.8130	21
Case 3	$\lambda^{0.5}$	$x_1 = (4.8342, 5.0000, 5.2254), x_2 = (10.9886, 11.0000, 11.2472)$	224.4106	210.2351	12.1189	22.9060	20
Case 4	$\begin{cases} \lambda & 0.3 \leq \lambda \leq 1 \\ 0 & 0 \leq \lambda < 0.3 \end{cases}$	$x_1 = (4.5238, 4.9951, 5.2401), x_2 = (10.7110, 11.0000, 11.3673)$	224.5178	193.5798	10.3691	20.5310	21

Table 5

Computation results of Test 3

	Optimization solutions	$Y_1$	$Y_2$	CD	CT	C
1	$x_1 = (4.5100, 4.9951, 5.1611), x_2 = (10.8955, 11.0000, 11.2828)$	224.1051	222.0662	10.0664	21.5470	14
2	$x_1 = (4.8933, 5.0000, 5.0607), x_2 = (10.9749, 11.0000, 11.2311)$	224.5483	222.2854	9.3827	19.8750	13
3	$x_1 = (4.8844, 5.0000, 5.3914), x_2 = (10.6385, 11.0000, 11.1725)$	224.8849	222.4760	10.0166	17.3590	17
4	$x_1 = (4.8038, 5.0000, 5.1821), x_2 = (10.7043, 11.0000, 11.4831)$	224.8755	221.9763	10.0919	24.6250	16
5	$x_1 = (4.9157, 5.0000, 5.2617), x_2 = (10.7470, 11.0000, 11.1360)$	224.6761	222.4789	9.6900	24.0320	18
6	$x_1 = (4.5845, 5.0000, 5.4958), x_2 = (10.6656, 11.0000, 11.0513)$	224.4286	222.1919	10.3633	24.3900	16
7	$x_1 = (4.8672, 4.9902, 5.3289), x_2 = (10.8135, 11.0000, 11.1829)$	224.7948	222.2969	9.8394	25.3750	19
8	$x_1 = (4.7202, 4.9951, 5.4219), x_2 = (10.7028, 11.0000, 11.0075)$	224.4004	222.1757	10.0315	23.2500	19
9	$x_1 = (4.8632, 4.9951, 5.2407), x_2 = (10.6387, 11.0000, 11.0100)$	224.1772	222.1572	9.6592	24.9530	18
10	$x_1 = (4.8818, 5.0000, 5.2377), x_2 = (10.7529, 11.0000, 11.1897)$	224.6644	222.3695	9.7466	25.9070	16
AV	$x_1 = (4.7924, 4.9976, 5.2782), x_2 = (10.7534, 11.0000, 11.1747)$	224.5555	222.2474	9.8888	23.1313	16.6

convergence performance; (3) Though the computational complexity is a bit larger than that of conventional algorithms, the difference is not great under high-performance parallel computing environment, so BPO&ID-FGA has good practicability; (4) BPO&ID-FGA, with the features of good interpretability and strong operability, have very good structure.

Synthesizing the computation results above and the theoretical analysis of Section 7, we see that BPO&ID-FGA is of stronger robust and good convergence, and suitable for the optimization problems under uncertain environment.

## 9. Conclusion

In this paper, by distinguishing the principal indices and secondary indices and using the restriction and supplementation relation between them, we give a comparison method of fuzzy information based on synthesizing effect and a description method of fuzzy information based on principal indices transformation. By using the structural characteristics of fuzzy information and the essence of constraint, we propose an axiomatic system for the fuzzy inequity degree and give an instructive metric method for measuring the fuzzy inequity degree. We also propose a new kind of fuzzy genetic algorithm based on the principal operation and inequity degree for the general optimization problems with fuzzy coefficients, fuzzy variables and fuzzy constraints (denoted by BPO&ID-FGA, for short). We consider the convergence of our algorithm using Markov chain theory and analyze its performance through simulation. Our analysis indicate that our algorithm not only merges decision preferences effectively into the optimization process, but also possesses many interesting advantages such

as strong robust, faster convergence, less iterations and less chance of being trapped into premature states, so our algorithm can be applied to many fuzzy fields such as artificial intelligence, manufacturing and management and optimization control etc.

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